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Exact moving and stationary solutions of a generalized discrete nonlinear Schrödinger equation

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Abstract

We obtain exact moving and stationary, spatially periodic and localized solutions of a generalized discrete nonlinear Schrödinger equation. More specifically, we find two different moving periodic wave solutions and a localized moving pulse solution. We also address the problem of finding exact stationary solutions and, for a particular case of the model when stationary solutions can be expressed through the Jacobi elliptic functions, we present a two-point map from which all possible stationary solutions can be found. Numerically we demonstrate the generic stability of the stationary pulse solutions and also the robustness of moving pulses in long-term dynamics.

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1. Introduction

The discrete nonlinear Schrödinger (DNLS) equation occurs ubiquitously [1] throughout modern science. Most notable is the role it plays in understanding the propagation of electromagnetic waves in glass fibers and other optical waveguides [2] as well as in the temporal evolution of Bose–Einstein condensates [3]. One of the variants of the DNLS model is the celebrated Ablowitz–Ladik (AL) model [4] which is an integrable model. Another aspect which stands out in favor of the AL model is that, while most other DNLS models have stationary wave solutions [5], this model has moving wave solutions. Further, these moving waves avoid the discreteness energy barrier (the so-called Peierls–Nabarro (PN) barrier). These solutions have played a major role in the computational studies of the corresponding continuum NLS model [6] as well as in developing perturbation techniques [7]. It is clearly of great interest to consider different variants of the DNLS equation [8–10] and to try to obtain exact moving wave solutions [5, 11]. The existence of such solutions might help in discovering new integrable models and would also help in further developing perturbative techniques in DNLS-type equations. The purpose of this paper is to report on the existence of exact moving

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as well as stationary solutions in a generalized DNLS model with seven parameters. For finite lattices, we find two different periodic moving wave solutions while for the infinite lattice we find a localized moving pulse solution.

In a recent paper, Pelinovsky [9] has addressed the question of spatial discretization of the NLS equation with cubic nonlinearity:

$$\mathrm{i}\dot{u} + u_{xx} + 2|u|^2 u = 0. \tag{1}$$

While the *standard* choice for the DNLS equation is

$$\mathrm{i}\dot{u}_n + u_{n+1} + u_{n-1} - 2u_n + 2|u_n|^2 u_n = 0, \tag{2}$$

strictly speaking, there is no unique choice. Perhaps the only constraint on the corresponding discrete model is that in the continuum limit it should go over to the NLS equation (1). By demanding that the semi-discretization is symplectic and few other requirements, Pelinovsky [9] showed that if one writes the DNLS equation in the form

$$\mathrm{i}\dot{u}_n + u_{n+1} + u_{n-1} - 2u_n + f(u_{n-1}, u_n, u_{n+1}) = 0, \tag{3}$$

then the most general form for the nonlinear function f is given by

$$\begin{aligned} f = {}& \alpha_1|u_n|^2 u_n + \alpha_2|u_n|^2(u_{n+1} + u_{n-1}) + \alpha_3 u_n^2(\bar{u}_{n+1} + \bar{u}_{n-1}) \\ & + \alpha_4 u_n(|u_{n+1}|^2 + |u_{n-1}|^2) + \alpha_5 u_n(\bar{u}_{n+1}u_{n-1} + \bar{u}_{n-1}u_{n+1}) \\ & + \alpha_6 \bar{u}_n\left(u_{n+1}^2 + u_{n-1}^2\right) + \alpha_7 \bar{u}_n u_{n+1} u_{n-1} + \alpha_8(|u_{n+1}|^2 u_{n+1} + |u_{n-1}|^2 u_{n-1}) \\ & + \alpha_9\left(\bar{u}_{n-1}u_{n+1}^2 + \bar{u}_{n+1}u_{n-1}^2\right) + \alpha_{10}(|u_{n+1}|^2 u_{n-1} + |u_{n-1}|^2 u_{n+1}), \end{aligned} \tag{4}$$

where $\bar{u}$ represents complex conjugate, and the real-valued parameters $(\alpha_1, \ldots, \alpha_{10})$ satisfy the continuity constraint:

$$\alpha_1 + \alpha_7 + 2(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_8 + \alpha_9 + \alpha_{10}) = 2. \tag{5}$$

The main goal of this paper is to obtain moving as well as stationary solutions in this generalized model and study their stability.

We note in passing that, under weaker constraints than that used in [9], one can add to (4) a term proportional to $u_n(|u_{n-1}u_n| + |u_n u_{n+1}|)$, which was demonstrated to be translationally invariant and conserving the norm, $\Sigma|u_n|^2$ [8].

The paper is organized as follows. In section 2, we derive exact moving solutions for a seven-parameter DNLS model of equation (4) with $\alpha_1 = \alpha_8 = 0$ under the constraint (5). In addition, for a five-parameter translationally invariant DNLS equation we obtain a nonlinear map from which all possible stationary solutions can be derived. In section 3, we present numerical results for the stationary and moving pulse solutions to demonstrate their stability. Section 4 summarizes our main findings and concludes the paper. In the appendix, we list the identities for the Jacobi elliptic functions used in the derivation of the periodic wave solutions.

2. Analytical results

We now show that two moving periodic wave solutions can be obtained with this general cubic polynomial in case terms of the type $|u_n|^2 u_n$ are absent, i.e.

$$\alpha_1 = \alpha_8 = 0. \tag{6}$$

It may be added here that the famous AL moving wave solutions are obtained in case only α_2 is nonzero while all other α_i are zero.

2.1. Solution I

In particular, it is not difficult to show that one of the exact periodic wave solutions to equation (3) [with f being given by equation (4) satisfying constraints (5) and (6)] is given by

$$u_n = A\exp[-\mathrm{i}(\omega t - kn + \delta)]\,\mathrm{dn}[\beta(n - vt + c), m], \tag{7}$$

provided the following six relations are satisfied:

$$v\beta = 2A^2(\alpha_2 - \alpha_3)\sin(k)\,\mathrm{cs}(\beta, m), \tag{8}$$

$$\alpha_6\,\mathrm{cs}(\beta, m)\sin(2k) + [\alpha_9\sin(3k) - \alpha_{10}\sin(k)]\,\mathrm{cs}(2\beta, m) = 0, \tag{9}$$

$$\begin{aligned}\frac{\sin(k)}{A^2} = {} & (\alpha_2 - \alpha_3)\sin(k)\,\mathrm{cs}^2(\beta, m) - \alpha_6\sin(2k)\,\mathrm{ds}(\beta, m)\,\mathrm{ns}(\beta, m)\\ & - [\alpha_9\sin(3k) - \alpha_{10}\sin(k)][\mathrm{cs}^2(2\beta, m) + \mathrm{ds}(2\beta, m)\,\mathrm{ns}(2\beta, m)],\end{aligned} \tag{10}$$

$$[\alpha_4 + \alpha_6\cos(2k)]\,\mathrm{cs}(\beta, m) + [\alpha_9\cos(3k) + \alpha_{10}\cos(k)]\,\mathrm{cs}(2\beta, m) = 0, \tag{11}$$

$$\begin{aligned}\frac{\cos(k)}{A^2} = {} & [\alpha_2 + \alpha_3]\cos(k)\,\mathrm{cs}^2(\beta, m) - [\alpha_4 + \alpha_6\cos(2k)]\,\mathrm{ds}(\beta, m)\,\mathrm{ns}(\beta, m)\\ & + [2\alpha_5\cos(2k) + \alpha_7]\,\mathrm{cs}(\beta, m)\,\mathrm{cs}(2\beta, m) - [\alpha_9\cos(3k) + \alpha_{10}\cos(k)]\\ & \times [\mathrm{ds}(2\beta, m)\,\mathrm{ns}(2\beta, m) - \mathrm{cs}^2(2\beta, m)],\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\omega}{A^2} - \frac{2}{A^2} = {} & -2[\alpha_2 + \alpha_3]\cos(k)\,\mathrm{ds}(\beta, m)\,\mathrm{ns}(\beta, m)\\ & + 2[\alpha_4 + \alpha_6\cos(2k)]\,\mathrm{cs}^2(\beta, m) - [2\alpha_5\cos(2k) + \alpha_7]\,\mathrm{cs}^2(\beta, m).\end{aligned} \tag{13}$$

Here c and δ are arbitrary constants, k, ω and v denote the wavenumber, frequency and velocity, respectively, of the periodic wave whereas $\mathrm{cs}(a, m)$, $\mathrm{ds}(a, m)$, $\mathrm{ns}(a, m)$ stand for the Jacobi elliptic functions $\mathrm{cn}(a, m)/\mathrm{sn}(a, m)$, $\mathrm{dn}(a, m)/\mathrm{sn}(a, m)$, $1/\mathrm{sn}(a, m)$ respectively with m being the modulus parameter ($0 \leqslant m \leqslant 1$) [12]. While deriving these relations, use has been made of the local identities (A.1)–(A.7) for Jacobi elliptic functions $\mathrm{dn}(x, m)$ [13] which are given in the appendix.

It may be noted that equations (8)–(13) determine the five parameters A, ω, k, v, β and give us one constraint between the eight parameters $\alpha_2, \ldots, \alpha_{10}$ (except α_8). In view of the constraint (5) between these parameters, it then follows that we have obtained a moving periodic wave solution with six parameters. As expected, in the limit $\alpha_2 \neq 0$ while all other $\alpha_i = 0$, we recover the well-known periodic wave solution of the AL problem [14]. Note that in order that the periodic solution be compatible with the lattice, the modulus m has to be chosen such that $\beta N_p = 2K(m)$ where $K(m)$ denotes the complete elliptic integral of the first kind [12] and N_p is the periodicity of the lattice [5].

2.2. Solution II

As in the AL case, there is another periodic wave solution to the DNLS equation (3) with f being given by equation (4) satisfying constraints (5) and (6). It is given by

$$u_n = A\sqrt{m}\exp[-\mathrm{i}(\omega t - kn + \delta)]\,\mathrm{cn}[\beta(n - vt + c), m], \tag{14}$$

provided the following six relations are satisfied

$$v\beta = 2A^2(\alpha_2 - \alpha_3)\sin(k)\,\mathrm{ds}(\beta, m), \tag{15}$$

$$\alpha_6\sin(2k)\,\mathrm{ds}(\beta, m) + [\alpha_9\sin(3k) - \alpha_{10}\sin(k)]\,\mathrm{ds}(2\beta, m) = 0, \tag{16}$$

$$\frac{\sin(k)}{A^2} = (\alpha_2 - \alpha_3)\sin(k)\,\mathrm{ds}^2(\beta,m) - \alpha_6\sin(2k)\,\mathrm{cs}(\beta,m)\,\mathrm{ns}(\beta,m) - [\alpha_9\sin(3k) - \alpha_{10}\sin(k)][\mathrm{ds}^2(2\beta,m) + \mathrm{cs}(2\beta,m)\,\mathrm{ns}(2\beta,m)], \tag{17}$$

$$[\alpha_4 + \alpha_6\cos(2k)]\,\mathrm{ds}(\beta,m) + [\alpha_9\cos(3k) + \alpha_{10}\cos(k)]\,\mathrm{ds}(2\beta,m) = 0, \tag{18}$$

$$\frac{\cos(k)}{A^2} = [\alpha_2 + \alpha_3]\cos(k)\,\mathrm{ds}^2(\beta,m) - [\alpha_4 + \alpha_6\cos(2k)]\,\mathrm{cs}(\beta,m)\,\mathrm{ns}(\beta,m) + [2\alpha_5\cos(2k) + \alpha_7]\,\mathrm{ds}(\beta,m)\,\mathrm{ds}(2\beta,m) - [\alpha_9\cos(3k) + \alpha_{10}\cos(k)] \times [\mathrm{cs}(2\beta,m)\,\mathrm{ns}(2\beta,m) - \mathrm{ds}^2(2\beta,m)], \tag{19}$$

$$\frac{\omega}{A^2} - \frac{2}{A^2} = -2[\alpha_2 + \alpha_3]\cos(k)\,\mathrm{cs}(\beta,m)\,\mathrm{ns}(\beta,m) + 2[\alpha_4 + \alpha_6\cos(2k)]\,\mathrm{ds}^2(\beta,m) - [2\alpha_5\cos(2k) + \alpha_7]\,\mathrm{ds}^2(\beta,m). \tag{20}$$

While deriving these relations, use has been made of the local identities (A.8)–(A.14) for the Jacobi elliptic function $\mathrm{cn}(x,m)$ [13] which have been given in the appendix.

As with the first solution, we again have a moving periodic wave solution with six parameters and again in the limit when only $\alpha_2 \neq 0$ while all other α_i are zero, we recover the well-known periodic wave solution of the AL problem [14]. In addition, note that in order that the periodic solution be compatible with the lattice, the modulus m has to be chosen such that $\beta N_p = 4K(m)$ where N_p is the periodicity of the lattice [5].

2.3. *Two-point maps for stationary solutions*

With the ansatz $u_n(t) = f_n\,\mathrm{e}^{-\mathrm{i}\omega t}$ we obtain from the DNLS equations (3) and (4) the following second-order difference equation for the amplitudes:

$$f_{n-1} - (2-\omega)f_n + f_{n+1} + \alpha_1 f_n^3 + (\alpha_2+\alpha_3)f_n^2(f_{n-1}+f_{n+1}) + (\alpha_4+\alpha_6)f_n\left(f_{n-1}^2 + f_{n+1}^2\right) + (2\alpha_5+\alpha_7)f_{n-1}f_nf_{n+1} + \alpha_8\left(f_{n-1}^3 + f_{n+1}^3\right) + (\alpha_9+\alpha_{10})f_{n-1}f_{n+1}(f_{n-1}+f_{n+1}) = 0. \tag{21}$$

For the following choice of parameters [that already includes the continuity constraint (5)]

$$\alpha_1 = \alpha_8 = 0, \quad \alpha_4 = -\alpha_6, \quad \alpha_9 = -\alpha_{10}, \qquad \text{and} \qquad \alpha_7 + 2[\alpha_2+\alpha_3+\alpha_5] = 2, \tag{22}$$

we get from (21) the following second-order difference equation for the amplitudes

$$f_{n-1} - (2-\omega)f_n + f_{n+1} + (\alpha_2+\alpha_3)f_n^2(f_{n-1}+f_{n+1}) + (2\alpha_5+\alpha_7)f_{n-1}f_nf_{n+1} = 0. \tag{23}$$

In this case, the stationary problem is exactly solvable. Indeed, one can obtain the first integral of (23) and present it in the form of a two-point nonlinear map

$$f_{n+1} = (2-\omega)\frac{Zf_n \pm \sqrt{R(f_n)}}{2-\omega+Yf_n^2}, \qquad R(f_n) = -\frac{Y}{2-\omega}\left(K - Xf_n^2 + f_n^4\right), \tag{24}$$

where

$$Z = \frac{(2-\omega)^2 - K(2\alpha_5+\alpha_7)^2}{2K(\alpha_2+\alpha_3)(2\alpha_5+\alpha_7) + 2(2-\omega)}, \qquad Y = 2(\alpha_2+\alpha_3)Z + (2\alpha_5+\alpha_7), \qquad X = -\frac{KY^2 + (2-\omega)^2(1-Z^2)}{(2-\omega)Y}. \tag{25}$$

Apart from the model parameters α_i and frequency ω, the nonlinear map (24), (25) contains the integration constant K. Due to the symmetry of equation (23), one can substitute f_{n+1} for f_{n-1} in (24). For any set of admissible values f_0, K and ω one can find the amplitudes of a stationary solution by iterating (24). For $R(f_n) > 0$, the map (24) gives two values for f_{n+1} and one should take the one which satisfies the original three-point problem (23). It is sufficient to take f_{n+1} different from f_{n-1}.

The above two-point map can also be constructed from the Jacobi elliptic function solutions (7) or (14) as described in our recent work on a discrete ϕ^4 model [15]. The corresponding DNLS equation has five free parameters because (22) sets up five constraints between the ten parameters (α_i) of the model. We note that *any* stationary solution to the DNLS equation defined by (3) and (4) with the parameters satisfying (22) can be constructed from the nonlinear map (24), (25). Such investigations have been carried out in our recent work on the DNLS equation [11] and the ϕ^4 equation [15, 16].

It is also worth pointing out that the three-point problem given by equation (23) and the three-point problem studied by Quispel *et al* [17] both can be presented in the following general form:

$$f_{n+1} = \frac{h_1(f_n) - h_2(f_n) f_{n-1}}{h_2(f_n) - h_3(f_n) f_{n-1}}. \tag{26}$$

For a particular choice of the functions $h_i(f_n)$, Quispel *et al* have found a two-point map (i.e., the first integral of the corresponding three-point problem) which is quadratic in both f_n and f_{n+1} [17]. For our choice of these functions,

$$h_1(f_n) = (2-\omega) f_n, \qquad h_2(f_n) = 1 + (\alpha_2+\alpha_3) f_n^2, \qquad h_3(f_n) = -(2\alpha_5+\alpha_7) f_n, \tag{27}$$

we found the map (24) which is, in general, quartic in f_n and quadratic in f_{n+1}. Clearly, our map (24) does not belong to the 12-parameter map discussed in [17].

The above result is new in that it generalizes the map reported in our recent work [15]. For completeness, let us also reproduce here the well-known result [9, 15] for the case of

$$\alpha_8 = \alpha_9 + \alpha_{10}, \quad \alpha_1 = \alpha_4 + \alpha_6, \quad \alpha_1 = 2\alpha_5 + \alpha_7, \quad \text{and} \quad 4\alpha_1 + 2[\alpha_2 + \alpha_3 + 2\alpha_8] = 2, \tag{28}$$

when the continuity constraint (5) is satisfied and (21) reduces to the following second-order difference equation:

$$\begin{aligned} f_{n-1} - (2-\omega) f_n + f_{n+1} + \alpha_1 f_n \left[f_{n-1}^2 + f_n^2 + f_{n+1}^2 + f_{n-1} f_{n+1} \right] + (\alpha_2+\alpha_3) \\ \times f_n^2 (f_{n-1} + f_{n+1}) + \alpha_8 \left[f_{n-1}^3 + f_{n+1}^3 + f_{n-1} f_{n+1} (f_{n-1} + f_{n+1}) \right] = 0. \end{aligned} \tag{29}$$

The first integral of (29) is

$$\begin{aligned} V(f_{n-1}, f_n) \equiv f_{n-1}^2 + f_n^2 - (2-\omega) f_{n-1} f_n + \alpha_1 \left(f_{n-1}^2 + f_n^2 \right) f_{n-1} f_n \\ + (\alpha_2+\alpha_3) f_{n-1}^2 f_n^2 + \alpha_8 \left(f_{n-1}^4 + f_n^4 \right) + K = 0, \end{aligned} \tag{30}$$

where K is the integration constant. This is so because (29) can be rewritten in the form

$$\frac{V(f_n, f_{n+1}) - V(f_{n-1}, f_n)}{f_{n+1} - f_{n-1}} = 0, \tag{31}$$

and one can verify that if $V(f_{n-1}, f_n) = 0$ then (29) is satisfied. Solving the algebraic problem (30) iteratively for an admissible initial value f_0 one can construct a stationary solution to (29). This model has six free parameters because (28) sets up four constraints between the ten parameters (α_i) of the model. In general, stationary solutions to the DNLS equation with the parameters satisfying (28) cannot be expressed in terms of the Jacobi elliptic functions, but, as was already mentioned, they can be constructed iteratively from (30) and they can be placed anywhere with respect to the lattice sites.

We note that the translationally invariant discrete models possessing the form of equation (31) have been introduced by Kevrekidis in [18].

2.4. Moving and stationary pulse solution

In the limit $m \to 1$, both the periodic moving wave solutions (7) and (14) reduce to the localized moving pulse solution

$$u_n = A\exp[-\mathrm{i}(\omega t - kn + \delta)]\,\mathrm{sech}[\beta(n - vt + c)], \tag{32}$$

and the relations (8)–(13) [as well as (15) to (20)] take a simpler form

$$v = \frac{2\sin(k)\sinh(\beta)}{\beta}, \tag{33}$$

$$2\alpha_6\sin(2k)\cosh(\beta) + \alpha_9\sin(3k) - \alpha_{10}\sin(k) = 0, \tag{34}$$

$$[\sinh^2(\beta) + (\alpha_3 - \alpha_2)A^2]\sin(k) = 0, \tag{35}$$

$$A^2 = \frac{2\sinh^2(\beta)\cosh(\beta)\cos(k)}{2(\alpha_2+\alpha_3)\cos(k)\cosh(\beta) + 2(\alpha_5-\alpha_6)\cos(2k) + \alpha_7 - 2\alpha_4}, \tag{36}$$

$$2[\alpha_4 + \alpha_6\cos(2k)]\cosh(\beta) + \alpha_9\cos(3k) + \alpha_{10}\cos(k) = 0, \tag{37}$$

$$\omega = 2[1 - \cos(k)\cosh(\beta)]. \tag{38}$$

From (33), pulse velocity is zero when $k = 0$ or $k = \pi$. In the former case, we have the non-staggered stationary pulse while in the latter case we have the staggered pulse. Needless to say that these remarks equally apply to the periodic wave solutions (7) and (14). In particular for $k = 0$ we obtain the non-staggered stationary pulse solution

$$\begin{aligned} &u_n = A\exp[-\mathrm{i}(\omega t + \delta)]\,\mathrm{sech}[\beta(n + c)],\\ &\omega = 2 - 2\cosh(\beta),\\ &A^2 = \frac{2\sinh^2(\beta)\cosh(\beta)}{2(\alpha_2+\alpha_3)\cosh(\beta) + 2(\alpha_5-\alpha_4-\alpha_6) + \alpha_7},\\ &2(\alpha_4+\alpha_6)\cosh(\beta) + \alpha_9 + \alpha_{10} = 0, \end{aligned} \tag{39}$$

while for $k = \pi$ we obtain the staggered stationary pulse:

$$\begin{aligned} &u_n = (-1)^n A\exp[-\mathrm{i}(\omega t + \delta)]\,\mathrm{sech}[\beta(n + c)],\\ &\omega = 2 + 2\cosh(\beta),\\ &A^2 = \frac{-2\sinh^2(\beta)\cosh(\beta)}{-2(\alpha_2+\alpha_3)\cosh(\beta) + 2(\alpha_5-\alpha_4-\alpha_6) + \alpha_7},\\ &2(\alpha_4+\alpha_6)\cosh(\beta) - \alpha_9 - \alpha_{10} = 0. \end{aligned} \tag{40}$$

3. Analysis of the pulse solution and numerical results

For given model parameters α_i, the moving pulse solution (32)–(38) is characterized by two parameters, $\beta > 0$ and $-\pi < k \leqslant \pi$. As can be seen from (33) and (38), the pulse velocity and frequency do not depend on model parameters while the pulse amplitude does, see (36). Using (33) one can express ω in (38) as a function of v and β. Also using (38) one can express the group velocity $\mathrm{d}\omega/\mathrm{d}k$. The pulse solution exists for given β and k if the right-hand side of (36) is positive and if (34), (35) and (37) can be satisfied together with the continuity constraint (5), where we assume (6).

As for the stationary pulse solution (39) or (40), for given model parameters α_i, the moving pulse solution is characterized by a single parameter $\beta > 0$. In general, as far as

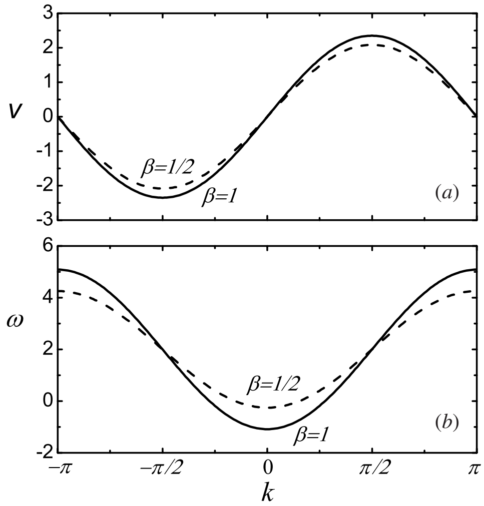

Figure 1. (a) Velocity v and (b) frequency ω of the pulse as functions of the wavenumber parameter k at a fixed value of the other parameter, inverse width of the pulse, $\beta = 1/2$ (dashed lines) and $\beta = 1$ (solid lines). These functions are defined by (33) and (38) and they do not depend on the model parameters α_i. Pulse velocity is zero at $k = 0$ and $k = \pi$, the former case corresponds to the non-staggered stationary pulse (39) while the latter case to the staggered stationary pulse (40).

the model parameters are fixed, the parameter β of the stationary pulse is also fixed through the last equation in (39) or (40). However, for $\alpha_4 = -\alpha_6$ and $\alpha_9 = -\alpha_{10}$, this constraint disappears and β can change continuously within a domain where $A^2 > 0$. Recall that in this particular case the stationary pulse solution can also be constructed from the two-point map presented in section 2.3, for which one should set the integration constant $K = 0$.

3.1. Different moving solutions

Coming back to the moving pulse solution (32)–(38), several comments are in order.

(1) The relations (33) and (38) are exactly the same as in the AL case [4]. It is indeed remarkable that the velocity v and the frequency ω in our case are identical to those in the AL model even though our model has eight nonlinear terms (with coefficients α_2 to α_{10} with $\alpha_8 = 0$) while AL has only one term with $\alpha_2 = 1$. It is amusing to note that these two relations have also been obtained by Pelinovsky and Rothos from an entirely different approach [19], namely from the linear dispersion relation for the corresponding differential advance-delay equation. In figure 1, we show how v and ω depend on one of the pulse parameter, k, at fixed values of the other parameter, $\beta = 1/2$ (dashed lines) and $\beta = 1$ (solid lines).

(2) Unfortunately, we do not know the Hamiltonian from which the DNLS equation (3) with f given by equation (4) can be derived. As a result, we cannot demonstrate the absence of the PN barrier from the energy consideration. However, since our stationary solutions have an effective translational invariance (i.e. the solution is valid for any value of the constant c), this suggests that the PN barrier would be zero for these solutions.

(3) From equation (35) it follows that the moving pulse solution exists only if α_2 and/or α_3 are nonzero. Further, in case $\alpha_2 = 0$, then it follows from equation (35) that $\alpha_3 < 0$.

(4) In the limit when only α_2 is nonzero while all other α_i are zero, we recover the well-known AL moving pulse solution [4].

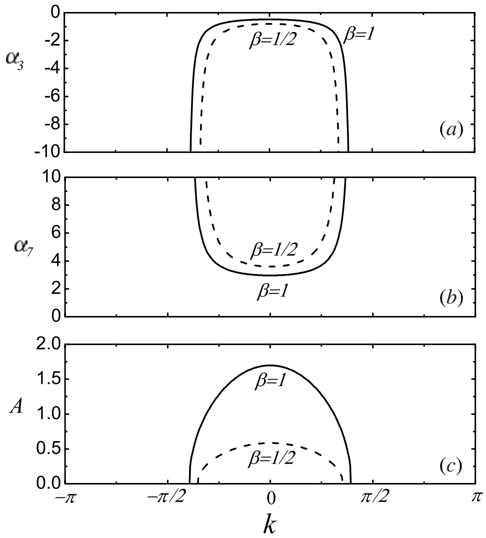

Figure 2. Nonzero model parameters α_3 and α_7 and the pulse amplitude A as functions of the parameter k at a fixed value of the other parameter $\beta = 1/2$ (dashed lines) and $\beta = 1$ (solid lines). These functions are defined by (43). For $\beta = 1/2$, the solution exists (i.e. A is real) for $|k| < 1.11$ while for $\beta = 1$ it exists for $|k| < 1.23$. The velocity v and frequency ω of the pulse are shown in figure 1.

(5) The sn-type and hence dark soliton solution can also be obtained in this generalized model provided the right-hand side of the continuity equation (5) is -2 (instead of 2).

(6) In case only α_2 and/or α_3 are nonzero while all other $\alpha_i = 0$, or if equation (28) is satisfied, then the generalized DNLS equation (3) with f given by equation (4) conserves the momentum defined by

$$P = \mathrm{i}\sum_n (u_{n+1}\bar{u}_n - \bar{u}_{n+1}u_n). \tag{41}$$

On the other hand, in case only α_5 and/or α_7 are nonzero while all other $\alpha_i = 0$, then the generalized DNLS equation (3) with f given by equation (4) conserves the momentum defined by

$$P = \mathrm{i}\sum_n (u_{n+2}\bar{u}_n - \bar{u}_{n+2}u_n). \tag{42}$$

Expression (42) is similar to that introduced in [15] for the ϕ^4 discrete equation.

(7) From equations (33)–(38), it follows that the moving pulse solution is also possible when only two of the eight parameters are nonzero. For example, the moving pulse solution (32) exists in case α_3 and α_7 are nonzero while all other α_i are zero. While the relations (33) and (38) are always valid, the other relations and the constraint (5) take the form

$$\alpha_3 = \frac{1}{1 - 2\cos(k)\cosh(\beta)}, \qquad \alpha_7 = 2(1-\alpha_3),$$
$$A^2 = \frac{\cos(k)\cosh(\beta)\sinh^2(\beta)}{\alpha_3[\cos(k)\cosh(\beta) - 1] + 1}. \tag{43}$$

For a pair of pulse parameters, k and β, we find α_3 and then α_7 and A from (43) and present the result in figure 2 for $\beta = 1/2$ (dashed lines) and $\beta = 1$ (solid lines). For

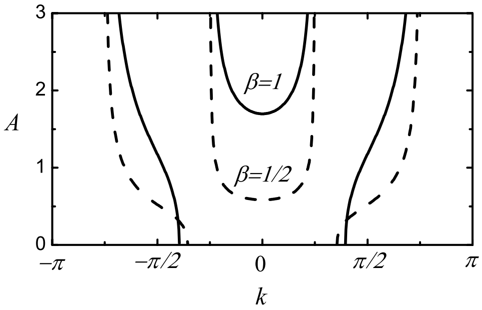

Figure 3. Pulse amplitude A as a function of the parameter k at $\beta = 1/2$ (dashed lines) and $\beta = 1$ (solid lines) for the model with three nonzero parameters, α_3, α_5 and α_7. Here we set $\alpha_5 = 1$ and find other model and pulse parameters from (44). For $\beta = 1/2$, the solution exists (i.e. A is real) for $|k| < \pi/4$ and $1.12 < |k| < 3\pi/4$, while for $\beta = 1$ it exists for $|k| < \pi/4$ and $1.24 < |k| < 3\pi/4$. The velocity v and frequency ω of the pulse are shown in figure 1.

$\beta = 1/2$, the solution exists (i.e. A is real) for $|k| < 1.11$ while for $\beta = 1$ it exists for $|k| < 1.23$. One can see that the non-staggered stationary pulse ($k = 0$) exists while the staggered stationary pulse ($k = \pi$) does not exist in this case.

(8) The moving pulse solution (32) also exists in case only (i) α_3 and α_5 are nonzero; (ii) α_2 and α_5 are nonzero and $\cos(2k) = 0$, i.e., regardless of the model parameters, in this case one can have only $k = \pm\pi/4$ and $k = \pm 3\pi/4$. Constraints similar to those in (43) are easily written down from relations (5) and (33)–(38). We were unable to find other sets of model parameters supporting the pulse solution when there are only two nonzero parameters.

(9) There are several possibilities, with three of the eight α_i being nonzero (the remaining five α_i being zero), in which case the moving pulse solution (32) is still valid. These cases are as follows: (i) α_2, α_3 and α_5 are nonzero; (ii) α_2, α_3 and α_7 are nonzero; (iii) α_3, α_5 and α_7 are nonzero; (iv) α_2, α_5 and α_7 are nonzero; (v) α_2, α_4 and α_6 are nonzero with $\alpha_4 = \alpha_6$ and $k = \pm\pi/2$; (vi) α_3, α_4 and α_6 are nonzero with $\alpha_4 = \alpha_6$ and $k = \pm\pi/2$; (vii) α_2, α_9 and α_{10} are nonzero with $\alpha_9 = \alpha_{10}$ and $k = \pm\pi/4$ or $k = \pm 3\pi/4$.

In all these cases, the constraints similar to those in (43) are easily obtained from relations (5) and (33)–(38). For example, in case only α_3, α_5 and α_7 are nonzero, while the relations (33) and (38) are always valid, the other relations and the constraint (5) take the form

$$\alpha_3 = \frac{1 - 2\alpha_5 \sin^2(k)}{1 - 2\cos(k)\cosh(\beta)}, \qquad \alpha_7 = 2(1 - \alpha_3 - \alpha_5),$$
$$A^2 = \frac{\cos(k)\sinh^2(\beta)\cosh(\beta)}{1 + \alpha_3[\cos(k)\cosh(\beta) - 1] - 2\alpha_5 \sin^2(k)}. \tag{44}$$

The number of constraints in this case is such that one has a free model parameter, say α_5, and pulse parameters k and β can change continuously within a certain domain. For α_5 with a small absolute value the solution is close to (43) shown in figure 2, but, for example, for $\alpha_5 = 1$ the result is qualitatively different, as it can be seen from figure 3. Also note that in this case the non-staggered stationary pulse ($k = 0$) exists while the staggered stationary pulse ($k = \pi$) does not exist.

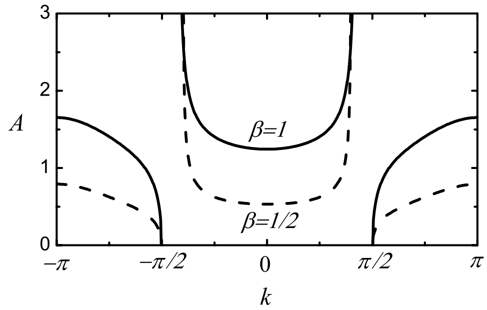

Figure 4. Pulse amplitude A as a function of the parameter k at $\beta = 1/2$ (dashed lines) and $\beta = 1$ (solid lines) for the model with three nonzero parameters, α_2, α_3 and α_5. The relation between model and pulse parameters is given by (45). For $\beta = 1/2$, the solution exists (i.e. A is real) for $|k| < 1.26$ and $\pi/2 < |k| \leqslant \pi$, while for $\beta = 1$ it exists for $|k| < 1.32$ and $\pi/2 < |k| \leqslant \pi$. The velocity v and frequency ω of the pulse are shown in figure 1.

On the other hand, in case only $\alpha_2, \alpha_3, \alpha_5$ are nonzero we have the following constraints

$$\begin{aligned} \alpha_3 &= \frac{-\alpha_5\cos(2k)}{2\cos(k)\cosh(\beta)}, \qquad \alpha_2 = 1 - \alpha_3 - \alpha_5, \\ A^2 &= \frac{\cos(k)\sinh^2(\beta)\cosh(\beta)}{(\alpha_2+\alpha_3)\cos(k)\cosh(\beta) + \alpha_5\cos(2k)}. \end{aligned} \tag{45}$$

The relation between pulse parameters and model parameters in this case is shown in figure 4. In this case, one has both non-staggered and staggered stationary pulse solutions for $k = 0$ and $k = \pi$, respectively.

We give two more solutions, for the case when only α_2, α_4 and α_6 are nonzero,

$$A^2 = \frac{\sinh^2\beta}{\alpha_2}, \qquad \alpha_4 = \frac{1-\alpha_2}{2}, \qquad \alpha_4 = \alpha_6, \qquad k = \pm\frac{\pi}{2}, \tag{46}$$

and for the case when only α_2, α_9 and α_{10} are nonzero,

$$A^2 = \frac{\sinh^2\beta}{\alpha_2}, \qquad \alpha_{10} = \frac{1-\alpha_2}{2}, \qquad \alpha_9 = \alpha_{10}, \qquad k = \pm\frac{\pi}{4}, \quad \text{or} \quad k = \pm\frac{3\pi}{4}. \tag{47}$$

These two *moving* solutions are interesting because for them the relations (22) are violated. These models have one free parameter, for example, $\alpha_2 > 0$. Among the two pulse parameters, only β can change continuously, while k can assume only a few isolated discrete values that do not depend on model parameters α_i. For the cases when there are only three nonzero parameters, we were unable to find sets of model parameters supporting the pulse solution other than those described above.

(10) Similarly, there are several possibilities when less than eight parameters are nonzero and still the moving pulse solution (32) continues to exist and relations similar to those in equation (43) can easily be obtained in all these cases.

(11) Since the DNLS equation (3), (4) with any set of parameters α_i satisfying the continuity constraint (5) reduces to the same NLS equation (1), for a sufficiently wide (small β) and slow (small $|k|$) pulse, all the solutions given above are close and can be well approximated by the moving solution to the continuous NLS equation.

3.2. Stability of the pulse solution

Let us now discuss the small amplitude vibration spectrum for the lattice containing a stationary pulse in order to observe the peculiarities of the spectrum of the pulse in a translationally invariant lattice and to discuss the stability of the pulse. The vibrational spectrum was calculated following the methodology presented in [20] similar to the work in [11]. In brief, we consider a small complex perturbation of a stationary solution and substitute the ansatz $u_n(t) = [f_n + \varepsilon_n(t)]\,\mathrm{e}^{-\mathrm{i}\omega t}$ with $\varepsilon_n(t) = a_n(t) + \mathrm{i}b_n(t)$ into the DNLS equation (3), (4) and obtain a linear equation for $\varepsilon_n(t)$. Separating real and imaginary parts of the equation, we derive the following system

$$\begin{pmatrix} \dot{\mathbf{b}} \\ \dot{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{K} \\ \mathbf{J} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix}, \tag{48}$$

where vectors $\mathbf{a}$ and $\mathbf{b}$ contain a_n and b_n, respectively, while the nonzero coefficients of matrices $\mathbf{K}$ and $\mathbf{J}$ are given by

$$\begin{aligned} K_{n,n-1} &= 1 + (\alpha_2+\alpha_3) f_n^2 + (2\alpha_5+\alpha_7) f_n f_{n+1} + (\alpha_9+\alpha_{10})\big(f_{n+1}^2 + 2 f_{n-1} f_{n+1}\big), \\ K_{n,n} &= -(2-\omega) + 2(\alpha_2+\alpha_3) f_n (f_{n-1}+f_{n+1}) + (\alpha_4+\alpha_6)\big(f_{n-1}^2 + f_{n+1}^2\big) + (2\alpha_5+\alpha_7) f_{n-1} f_{n+1}, \\ K_{n,n+1} &= 1 + (\alpha_2+\alpha_3) f_n^2 + (2\alpha_5+\alpha_7) f_{n-1} f_n + (\alpha_9+\alpha_{10})\big(f_{n-1}^2 + 2 f_{n-1} f_{n+1}\big), \end{aligned} \tag{49}$$

$$\begin{aligned} J_{n,n-1} &= -1 - (\alpha_2-\alpha_3) f_n^2 - 2\alpha_6 f_{n-1} f_n - \alpha_7 f_n f_{n+1} - 2\alpha_9 f_{n-1} f_{n+1} + (\alpha_9-\alpha_{10}) f_{n+1}^2, \\ J_{n,n} &= (2-\omega) - 2\alpha_3 f_n (f_{n-1}+f_{n+1}) - (\alpha_4-\alpha_6)\big(f_{n-1}^2 + f_{n+1}^2\big) - (2\alpha_5-\alpha_7) f_{n-1} f_{n+1}, \\ J_{n,n+1} &= -1 - (\alpha_2-\alpha_3) f_n^2 - 2\alpha_6 f_n f_{n+1} - \alpha_7 f_{n-1} f_n - 2\alpha_9 f_{n-1} f_{n+1} + (\alpha_9-\alpha_{10}) f_{n-1}^2. \end{aligned} \tag{50}$$

A stationary solution is characterized as linearly stable if and only if the eigenvalue problem

$$\begin{pmatrix} 0 & \mathbf{K} \\ \mathbf{J} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} = \gamma \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} \tag{51}$$

results in nonpositive real parts of all eigenvalues γ.

Setting in the above matrices $f_n = 0$, and solving the resulting eigenvalue problem one finds the spectrum of vacuum

$$\Omega = \pm\left[-\omega + 4\sin^2\left(\frac{Q}{2}\right)\right], \tag{52}$$

where Ω and Q are the frequency and the wavenumber of a small-amplitude harmonic mode, respectively. A stationary pulse was placed in the middle of a lattice of $N = 200$ points and the eigenvalue problem (51) was solved employing periodic boundary conditions. Here we do not aim to present a comprehensive numerical study of the stability of the pulse because the DNLS equation under consideration has a multi-dimensional parameter space and such an exhaustive study would entail enormous effort. Instead, our intent is to check several sets of parameters and to provide a few examples illustrating the generic stability of the pulse solution.

Two examples of stationary, stable pulses and their spectra are presented in figure 5. Left panels present the results for a non-staggered pulse, while the right panels are for a staggered pulse. Model parameters correspond to a translationally invariant lattice, i.e., they satisfy (22). For panels (*a*) and (*b*), parameters are $\alpha_2 = 1, \alpha_3 = -1/2, \alpha_4 = -\alpha_6 = 1/2, \alpha_5 = -1/2, \alpha_7 = 2$ and $\alpha_9 = -\alpha_{10} = -1/2$. For panels ($a'$) and ($b'$), parameters are

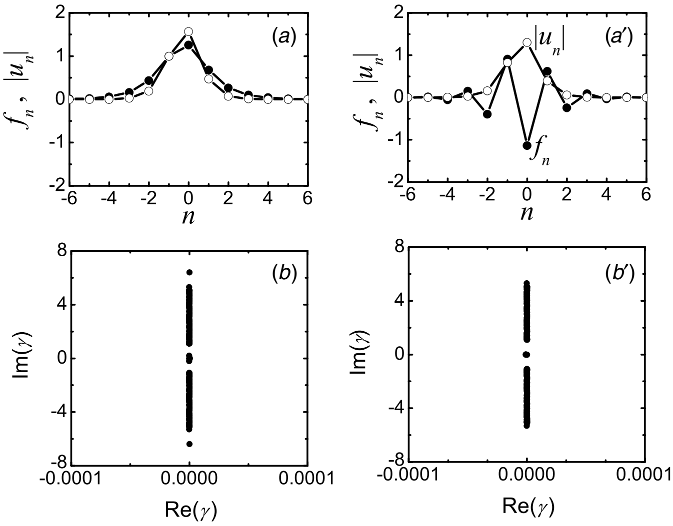

Figure 5. Two examples of (a) and (a') stationary pulse profiles and (b), (b') their spectra. Left panels show the results for a non-staggered pulse, while right panels are for a staggered pulse. Model parameters correspond to a translationally invariant lattice, i.e., they satisfy (22). For (a) and (b), parameters are $\alpha_2 = 1, \alpha_3 = -1/2, \alpha_4 = -\alpha_6 = 1/2, \alpha_5 = -1/2, \alpha_7 = 2$ and $\alpha_9 = -\alpha_{10} = -1/2$. For ($a'$) and ($b'$), parameters are $\alpha_2 = 1, \alpha_3 = 0, \alpha_4 = -\alpha_6 = 1/2, \alpha_5 = -1/2, \alpha_7 = 1$ and $\alpha_9 = -\alpha_{10} = -1/2$. The pulses are defined by, respectively, (39) and (40) with parameters $\beta = 1, \delta = 0$ and $c = 0.25$. Pulses are placed asymmetrically with respect to the lattice, nevertheless, they are stationary and stable since all eigenvalues γ have zero real parts. The spectra also contain two pairs of zero eigenvalues, one pair corresponds to the translational invariance and another to the invariance with respect to the phase shift.

$\alpha_2 = 1, \alpha_3 = 0, \alpha_4 = -\alpha_6 = 1/2, \alpha_5 = -1/2, \alpha_7 = 1$ and $\alpha_9 = -\alpha_{10} = -1/2$. The choice of parameters is rather arbitrary. For the non-staggered pulse, all coefficients are nonzero so that all terms of the DNLS equation are involved. For the staggered pulse, we found that nonzero α_3 makes the pulse unstable, that is why we set this coefficient equal to zero. The non-staggered and staggered pulses are defined by, respectively, (39) and (40) with parameters $\beta = 1, \delta = 0$ and $c = 0.25$. We then found for the non-staggered pulse $\omega = -1.0862$ and $A = 1.2946$, and for the staggered one, $\omega = 5.0862$ and $A = 1.1752$. The pulses are placed asymmetrically with respect to the lattice, nevertheless, they are stationary and stable since all eigenvalues γ have zero real parts. The spectrum of non-staggered pulse contains the spectrum of vacuum (52) with the bands $1.0862 \leqslant |\Omega| \leqslant 5.0862$; the three pulse internal modes with frequencies ± 0.195, ± 5.290 and ± 6.396; and the two pairs of zero eigenvalues, one pair corresponding to the translational invariance and another to the invariance with respect to the phase shift. The spectrum of staggered pulse is similar but it contains not three but only one pulse internal mode with frequencies ± 5.308.

We have checked the stability of stationary pulses (both non-staggered and staggered) with different $\beta \sim 1$, and also different positions with respect to the lattice, c, and for various model parameters with $|\alpha_i| \sim 1$, and in many of the cases found these pulses to be stable. Thus, we conclude that the stationary pulse solutions (39) and (40) to the DNLS equation (3), (4) with parameters satisfying (22) are generically stable.

We have also checked the stability of a stationary pulse in the model where the pulse solution exists only for a selected β and, for the pulse placed asymmetrically with respect to the lattice we found that it is stable. In this simulation, for the solution (39) we took the following

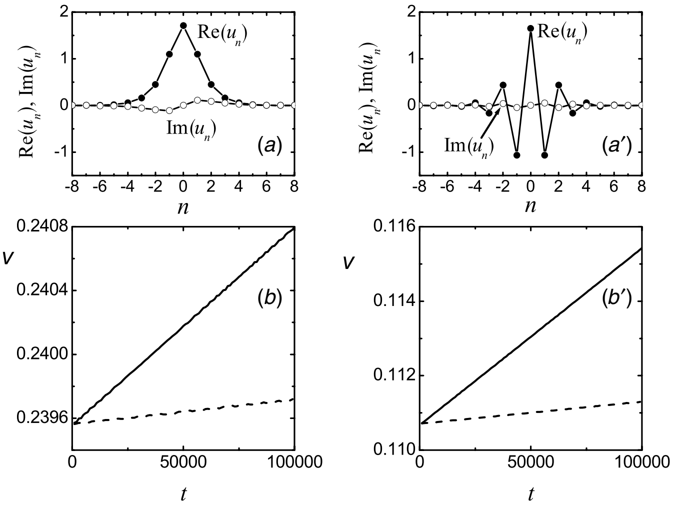

Figure 6. (a) Non-staggered moving pulse at $t = 0$ and (a') the same for the staggered pulse. In (b) and (b'), the long-term evolution of pulse velocity is shown for the corresponding pulses for the integration steps of $\tau = 5 \times 10^{-3}$ (solid line) and $\tau = 2.5 \times 10^{-3}$ (dashed line). A numerical scheme with an accuracy $O(\tau^4)$ is employed. Pulses preserve their velocity with the accuracy increasing with the increase in the accuracy of numerical integration. Within the numerical run, the pulse dynamics is stable in spite of the presence of small perturbations in the system in the form of rounding errors and integration scheme errors. The pulse in (a) is given by (32), (33), (38) and (44). The model and pulse parameters are as follows: $\alpha_3 = -0.473\,034, a_5 = 1$ and $\alpha_7 = 0.946\,068$; $\beta = 1, k = 0.102\,102$ (close to 0), $v = 0.239\,563, \omega = -1.070\,09$ and $A = 1.7087$. The pulse in (a') is given by (32), (33), (38) and (45). The model and pulse parameters are as follows: $\alpha_2 = 0.603\,116, \alpha_3 = 0.096\,8843$ and $a_5 = 0.3$; $\beta = 1, k = 3.094\,47$ (close to π), $v = 0.110\,719, \omega = 5.082\,74$ and $A = 1.651\,72$.

pulse parameters $\beta = 1/2, c = 1/4$, and model parameters $\alpha_2 = -0.2553, \alpha_4 = \alpha_6 = -1/2$ and $\alpha_9 = 2.2553$ with all other α_i equal to zero. We then found $\omega = -0.2553$ and $A = 0.6557$.

The robustness of *moving* pulse solutions was checked by observing the evolution of their velocity in a long-term numerical run. For pulses with amplitudes $A \sim 1$ and velocities $v \sim 0.1$ and for various model parameters supporting the pulse, $|\alpha_i| \sim 1$, we found that the pulse typically preserves its velocity with a high accuracy. Two examples of such simulations, one for the non-staggered pulse and another one for the staggered pulse are given in figures 6(a), (b) and (a'), (b'), respectively. In (a) and (a'), we show the pulse configuration at $t = 0$ and in (b) and (b') the pulse velocity as a function of time for two different integration steps, $\tau = 5 \times 10^{-3}$ (solid lines) and $\tau = 2.5 \times 10^{-3}$ (dashed lines), while the numerical scheme with an accuracy $O(\tau^4)$ is employed.

In both cases, one can note the linear increase in the pulse velocity with time, which is due to the numerical error, since the slope of the line decreases with the decrease in τ. The presence of perturbation in the form of rounding errors and integration scheme errors does not result in pulse instability within the numerical run. Velocity increase rate for the staggered pulse in (b') is larger than for the non-staggered one in (b). This can be easily understood because the frequency of the staggered pulse is almost five times larger than that of the non-staggered one.

The pulse presented in figure 6(a) is given by (44). The model has one free parameter and we set $\alpha_5 = 1$. For the pulse parameters, we set $\beta = 1$ and $k = 0.102\,102$ (close to zero). Then we find from (32), (33), (38) and (44) the pulse velocity $v = 0.239\,563$,

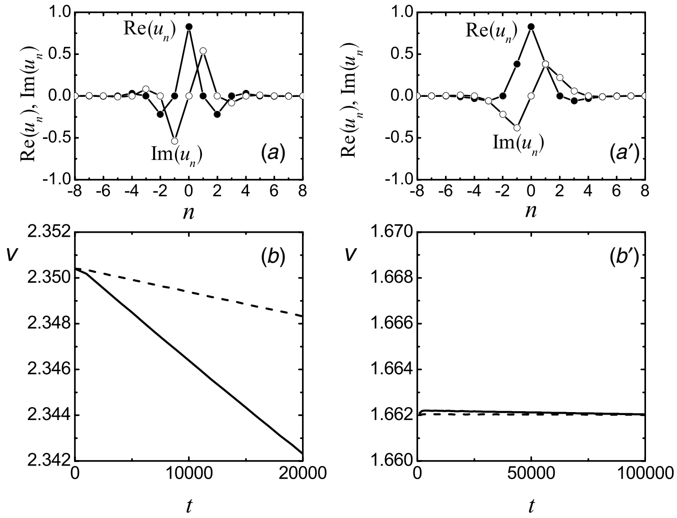

Figure 7. Results similar to that shown in figure 6 but for models that are not translationally invariant. Panels (*a*) and (a') show the moving pulse profiles at $t = 0$. In (*b*) and (b'), the long-term evolution of pulse velocity is shown for the corresponding pulses. The integration steps are (*b*) $\tau = 10^{-3}$ (solid line) and $\tau = 5 \times 10^{-4}$ (dashed line) and (b') $\tau = 5 \times 10^{-3}$ (solid line) and $\tau = 2.5 \times 10^{-3}$ (dashed line). The pulse in (*a*) is given by (32), (33), (38) and (46). The model and pulse parameters are as follows: $\alpha_2 = 2, a_4 = -1/2$ and $\alpha_6 = -1/2$; $\beta = 1, k = \pi/2, v = 1.662, \omega = -0.1822$ and $A = 0.8310$. The pulse in (a') is given by (32), (33), (38) and (47). The model and pulse parameters are as follows: $\alpha_2 = 2, \alpha_9 = -1/2$ and $a_{10} = -1/2$; $\beta = 1, k = \pi/4, v = 2.350, \omega = 2$ and $A = 0.8310$.

frequency $\omega = -1.070\,09$ and amplitude $A = 1.7087$, and the dependent model parameters $\alpha_3 = -0.473\,034$ and $\alpha_7 = 0.946\,068$.

In figure 6(a'), the moving pulse solution is given by (45). The model has one free parameter and we set $a_5 = 0.3$. For the pulse parameters, we set $\beta = 1$ and $k = 3.094\,47$ (close to π). Then we find from (32), (33), (38) and (45) the pulse velocity $v = 0.110\,719$, frequency $\omega = 5.082\,74$ and amplitude $A = 1.651\,72$, and the dependent model parameters $\alpha_2 = 0.603\,116$ and $\alpha_3 = 0.096\,8843$.

Similar results were observed for the cases when only α_3 and α_5 are nonzero; only α_3 and α_7 are nonzero; only α_2, α_3 and α_5 are nonzero; only α_2, α_3 and α_7 are nonzero; and only α_2, α_5 and α_7 are nonzero.

So far we have studied numerically the pulses in the models with the parameters satisfying (22). However, moving pulse solutions exist even in the case when (22) is violated. Two such solutions are presented by (46) and (47) together with (32), (33) and (38). As can be seen from figure 7, the pulses show a stable long-term dynamics with pulse velocity being practically constant with the accuracy increasing with decrease in the step size of numerical integration. The pulse in (*a*) is given by (32), (33), (38) and (46). The model and pulse parameters are as follows: $\alpha_2 = 2, a_4 = -1/2$ and $\alpha_6 = -1/2$; $\beta = 1, k = \pi/2, v = 1.662, \omega = -0.1822$ and $A = 0.8310$. The pulse in (a') is given by (32), (33), (38) and (47). The model and pulse parameters are as follows: $\alpha_2 = 2, \alpha_9 = -1/2$ and $a_{10} = -1/2$; $\beta = 1, k = \pi/4, v = 2.350, \omega = 2$ and $A = 0.8310$.

Velocity increase rate in (*b*) is considerably larger than in (b') (note the different abscissa scale for these two panels) and this result can be expected when we take into account that pulse frequency in (*b*) is 11 times larger than in (b').

4. Conclusions and future challenges

For the nine-parameter DNLS equation (3), (4) with the continuity constraint (5), in sections 2.1 and 2.2, we obtained the two moving periodic wave solutions for the case of $\alpha_1 = \alpha_8 = 0$ (thus, the moving solutions are supported by the seven-parameter model). The solutions have the form of dn and cn Jacobi elliptic functions. In the limit $m \to 1$, both solutions reduce to the moving pulse solution (see section 2.4). We found and described several sets of model parameters supporting the moving pulse solution. For the particular choice of model parameters (22), the problem of finding stationary solutions is integrable and the first integral of this problem was given in section 2.3 in the form of a nonlinear map. From this map *any* stationary solution of the corresponding problem can be constructed iteratively.

We found the stationary pulse solutions to be generically stable, i.e., for rather arbitrary choice of model parameters $|\alpha_i| \sim 1$, in many cases, the spectra of the small-amplitude vibrations calculated for the lattice containing a pulse included no eigenvalues with positive real parts. In addition, we confirmed the robustness of moving pulses by observing the pulse velocity evolution in a long-term numerical run. We found the velocity to be nearly constant and the deviation from constancy was attributed to the influence of the accuracy of the numerical integration. We specifically note that the moving pulse solutions exist and they exhibit a stable behavior in long-term numerical runs even for models which do not support translationally invariant stationary pulse solutions, as demonstrated in figure 7.

On using the identities for the Jacobi elliptic functions cn and dn given below and similar identities for sn, one can similarly obtain exact solutions of a rather general discrete $\lambda\phi^4$ field theory with four parameters, as well as of a modified Fermi–Pasta-Ulam (FPU) model [21], which will be discussed elsewhere. Our results are potentially important for optical pulse propagation in glass fibers and optical waveguides [2] and time evolution of Bose–Einstein condensates [3].

Acknowledgments

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Appendix

We list here the various identities for the Jacobi elliptic functions $\mathrm{dn}(x, m)$ and $\mathrm{cn}(x, m)$ which have been used in obtaining the various solutions in this paper.

Identities for $\mathrm{dn}(x, m)$

$$\begin{aligned}\mathrm{dn}^2(x, m)[\mathrm{dn}(x+a, m) + \mathrm{dn}(x-a, m)]\\ = -\mathrm{cs}^2(a, m)[\mathrm{dn}(x+a, m) + \mathrm{dn}(x-a, m)] + 2\mathrm{ns}(a, m)\,\mathrm{ds}(a, m)\,\mathrm{dn}(x, m),\end{aligned} \tag{A.1}$$

$$\begin{aligned}\mathrm{dn}(x, m)\,\mathrm{dn}(x+a, m)\,\mathrm{dn}(x-a, m) = -\mathrm{cs}(a, m)\,\mathrm{cs}(2a, m)[\mathrm{dn}(x+a, m)\\ + \mathrm{dn}(x-a, m)] + \mathrm{cs}^2(a, m)\,\mathrm{dn}(x, m),\end{aligned} \tag{A.2}$$

$$\begin{aligned}\mathrm{dn}(x, m)[\mathrm{dn}^2(x+a, m) + \mathrm{dn}^2(x-a, m)] = \mathrm{ds}(a, m)\,\mathrm{ns}(a, m)[\mathrm{dn}(x+a, m) + \mathrm{dn}(x-a, m)]\\ - 2\mathrm{cs}^2(a, m)\,\mathrm{dn}(x, m) + m\,\mathrm{cs}(a, m)[\mathrm{cn}(x+a, m)\,\mathrm{sn}(x+a, m)\\ - \mathrm{cn}(x-a, m)\,\mathrm{sn}(x-a, m)],\end{aligned} \tag{A.3}$$

$$\begin{aligned}\mathrm{dn}(x+a,m)\,\mathrm{dn}(x-a,m)&[\mathrm{dn}(x+a,m)+\mathrm{dn}(x-a,m)]\\&=[\mathrm{ds}(2a,m)\,\mathrm{ns}(2a,m)-\mathrm{cs}^2(2a,m)][\mathrm{dn}(x+a,m)+\mathrm{dn}(x-a,m)]\\&\quad+m\mathrm{cs}(2a,m)[\mathrm{cn}(x+a,m)\,\mathrm{sn}(x+a,m)-\mathrm{cn}(x-a,m)\,\mathrm{sn}(x-a,m)],\end{aligned}\tag{A.4}$$

$$\begin{aligned}\mathrm{dn}^2(x,m)[\mathrm{dn}(x+a,m)-\mathrm{dn}(x-a,m)]&=-\mathrm{cs}^2(a,m)[\mathrm{dn}(x+a,m)-\mathrm{dn}(x-a,m)]\\&\quad-2m\mathrm{cs}(a,m)\,\mathrm{cn}(x,m)\,\mathrm{sn}(x,m),\end{aligned}\tag{A.5}$$

$$\begin{aligned}\mathrm{dn}(x,m)[\mathrm{dn}^2(x+a,m)-\mathrm{dn}^2(x-a,m)]&=\mathrm{ds}(a,m)\,\mathrm{ns}(a,m)[\mathrm{dn}(x+a,m)-\mathrm{dn}(x-a,m)]\\&\quad+m\mathrm{cs}(a,m)[\mathrm{cn}(x+a,m)\,\mathrm{sn}(x+a,m)+\mathrm{cn}(x-a,m)\,\mathrm{sn}(x-a,m)],\end{aligned}\tag{A.6}$$

$$\begin{aligned}\mathrm{dn}(x+a,m)\,\mathrm{dn}(x-a,m)&[\mathrm{dn}(x+a,m)-\mathrm{dn}(x-a,m)]\\&=[\mathrm{ds}(2a,m)\,\mathrm{ns}(2a,m)+\mathrm{cs}^2(2a,m)][\mathrm{dn}(x+a,m)-\mathrm{dn}(x-a,m)]\\&\quad+m\mathrm{cs}(2a,m)[\mathrm{cn}(x+a,m)\,\mathrm{sn}(x+a,m)+\mathrm{cn}(x-a,m)\,\mathrm{sn}(x-a,m)].\end{aligned}\tag{A.7}$$

Identities for $\mathrm{cn}(x,m)$

$$\begin{aligned}m\mathrm{cn}^2(x,m)[\mathrm{cn}(x+a,m)+\mathrm{cn}(x-a,m)]&=-\mathrm{ds}^2(a,m)[\mathrm{cn}(x+a,m)+\mathrm{cn}(x-a,m)]\\&\quad+2\mathrm{ns}(a,m)\,\mathrm{cs}(a,m)\,\mathrm{cn}(x,m),\end{aligned}\tag{A.8}$$

$$\begin{aligned}m\mathrm{cn}(x,m)\,\mathrm{cn}(x+a,m)\,\mathrm{cn}(x-a,m)&=-\mathrm{ds}(a,m)\,\mathrm{ds}(2a,m)[\mathrm{cn}(x+a,m)\\&\quad+\mathrm{cn}(x-a,m)]+\mathrm{ds}^2(a,m)\,\mathrm{cn}(x,m),\end{aligned}\tag{A.9}$$

$$\begin{aligned}m\mathrm{cn}(x,m)[\mathrm{cn}^2(x+a,m)+\mathrm{cn}^2(x-a,m)]&=\mathrm{cs}(a,m)\,\mathrm{ns}(a,m)[\mathrm{cn}(x+a,m)+\mathrm{cn}(x-a,m)]\\&\quad-2\,\mathrm{ds}^2(a,m)\,\mathrm{cn}(x,m)+\mathrm{ds}(a,m)[\mathrm{dn}(x+a,m)\,\mathrm{sn}(x+a,m)\\&\quad-\mathrm{dn}(x-a,m)\,\mathrm{sn}(x-a,m)],\end{aligned}\tag{A.10}$$

$$\begin{aligned}m\mathrm{cn}(x+a,m)\,\mathrm{cn}(x-a,m)&[\mathrm{cn}(x+a,m)+\mathrm{cn}(x-a,m)]\\&=[\mathrm{cs}(2a,m)\,\mathrm{ns}(2a,m)-\mathrm{ds}^2(2a,m)][\mathrm{cn}(x+a,m)+\mathrm{cn}(x-a,m)]\\&\quad+\mathrm{ds}(2a,m)[\mathrm{dn}(x+a,m)\,\mathrm{sn}(x+a,m)-\mathrm{dn}(x-a,m)\,\mathrm{sn}(x-a,m)],\end{aligned}\tag{A.11}$$

$$\begin{aligned}m\mathrm{cn}^2(x,m)[\mathrm{cn}(x+a,m)-\mathrm{cn}(x-a,m)]&=-\mathrm{ds}^2(a,m)[\mathrm{cn}(x+a,m)-\mathrm{cn}(x-a,m)]\\&\quad-2\,\mathrm{ds}(a,m)\,\mathrm{dn}(x,m)\,\mathrm{sn}(x,m),\end{aligned}\tag{A.12}$$

$$\begin{aligned}m\mathrm{cn}(x,m)[\mathrm{cn}^2(x+a,m)-\mathrm{cn}^2(x-a,m)]&=\mathrm{cs}(a,m)\,\mathrm{ns}(a,m)[\mathrm{cn}(x+a,m)-\mathrm{cn}(x-a,m)]\\&\quad+\mathrm{ds}(a,m)[\mathrm{dn}(x+a,m)\,\mathrm{sn}(x+a,m)+\mathrm{dn}(x-a,m)\,\mathrm{sn}(x-a,m)],\end{aligned}\tag{A.13}$$

$$\begin{aligned}m\mathrm{cn}(x+a,m)\,\mathrm{cn}(x-a,m)&[\mathrm{cn}(x+a,m)-\mathrm{cn}(x-a,m)]\\&=[\mathrm{cs}(2a,m)\,\mathrm{ns}(2a,m)+\mathrm{ds}^2(2a,m)][\mathrm{cn}(x+a,m)-\mathrm{cn}(x-a,m)]\\&\quad+\mathrm{ds}(2a,m)[\mathrm{dn}(x+a,m)\,\mathrm{sn}(x+a,m)+\mathrm{dn}(x-a,m)\,\mathrm{sn}(x-a,m)].\end{aligned}\tag{A.14}$$

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